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Besov Spaces and the Multifractal Hypothesis

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Parisi and Frisch proposed some time ago an explanation for "multiscaling" of turbulent velocity structure functions in terms of a "multifractal hypothesis," i.e., they conjectured that the velocity field has local Hölder exponents in a range $[h_{\min}, h_{\max}]$, with exponents $< h$ occurring on a set $S(h)$ with a fractal dimension $D(h)$. Heuristic reasoning led them to an expression for the scaling exponent z_p of pth order as the Legendre transform of the codimension $d - D(h)$. We show here that a part of the multifractal hypothesis is correct under even weaker assumptions: namely, if the velocity field has L^p-mean Hölder index s, i.e., if it lies in the Besov space $B_p^{s,\infty}$, then local Hölder regularity is satisfied. If $s < d/p$, then the hypothesis is true in a generalized sense of Hölder space with negative exponents and we discuss the proper definition of local Hölder classes of negative index. Finally, if a certain "box-counting dimension" exists, then the Legendre transform of its codimension gives the scaling exponent z_p, and, more generally, the maximal Besov index of order p, as $s_p = z_p/p$. Our method of proof is derived from a recent paper of S. Jaffard using compactly-supported, orthonormal wavelet bases and gives an extension of his results. We discuss implications of the theorems for ensemble-average scaling and fluid turbulence.

KEY WORDS: Multifractals; Besov spaces; turbulence.

1. INTRODUCTION

In a short Appendix to a longer article,[1] Parisi and Frisch in 1985 proposed an intuitive explanation of the scaling laws for velocity structure functions reported in experimental work of Anselmet *et al.*[2] The basic phenomenon observed was a "multiscaling" property, in which pth-order moments of velocity differences exhibited short-distance scaling but with exponents depending nonlinearly on p, that is, as $l \to 0$,

$$\langle |\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})|^p \rangle \sim l^{z_p} \tag{1}$$

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with $z_p \neq (\text{const})p$. In this relation, $\sim$ is understood to mean that the logarithm of the left side divided by $\log l$ has the limit z_p as $l \to 0$. To explain these observations, Parisi and Frisch conjectured that the turbulent velocity field in the zero-viscosity limit has a spectrum of Hölder singularities $[h_{\min}, h_{\max}]$ and that the set $S(h)$ on which the Hölder exponent is $<h$ has a fractal dimension $D(h)$. Heuristically, it then follows that

$$\langle (\delta v(l))^p \rangle \sim \int \mu(dh)\, l^{ph + (d - D(h))}$$
$$\sim l^{z_p} \tag{2}$$

for $l \to 0$, with

$$z_p = \inf_h [ph + (d - D(h))] \tag{3}$$

In these formulas d is the space dimension, $d = 3$ in the realistic case. The mean $\langle \cdot \rangle$ may be interpreted variously as space, time, or ensemble averaging.

Both the experimental observations which motivated the Parisi–Frisch theory and their "multifractal" explanation of the phenomena have remained controversial. For example, some questions have been raised[3] about the data analysis procedures of Anselmet *et al.* Also, the Parisi–Frisch explanation was considered by the authors themselves as being just one possible explanation of the "multiscaling" properties. We wish to distinguish here two different issues. We shall refer to the Parisi–Frisch interpretation of multiscaling as the "multifractal hypothesis." If the expectations in the scaling laws are considered as space averages, then the validity of this hypothesis is essentially a question of abstract function theory. However, even if it is true, there is still the question whether the multiscaling indicated experimentally really exists for turbulent velocity fields in the zero-viscosity limit and therefore whether they are indeed multifractal functions. We shall refer to the thesis that inertial-range velocity fields are multifractal functions as the "multifractal model" of turbulence. Unlike the previous issue, the validity of the "multifractal model" depends upon dynamical and statistical properties of the Navier–Stokes equations in the zero-viscosity limit.

What we shall show in this work is that, perhaps surprisingly, part of the "multifractal hypothesis" is a necessary consequence of the multiscaling. More precisely, we shall show that if a velocity field obeys the scaling law, as $l \to 0$,

$$\langle |\mathbf{v}(\mathbf{r} + \mathbf{l}) - \mathbf{v}(\mathbf{r}))|^p \rangle \sim l^{z} \tag{4}$$

for *any single* $p \geqslant 1$ with absolute values of differences and with the expectation considered as a space average, then it is indeed a multifractal function in the sense that it has local Hölder regularity, with Hölder index h occurring on a space set $S(h)$. [We should properly refer to this as multifractality in a "weak sense," since it is not required that the sets $S(h)$ be at all fractal in nature.] In fact, we shall show that this "weak multifractality" is a consequence of just the property of L^p-mean Hölder continuity, i.e.,

$$\langle |\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r}))|^p \rangle^{1/p} \leqslant C \cdot l^s \tag{5}$$

for all $\mathbf{l}$ with $l \leqslant 1$, for some index $s > 0$ and some constant $C > 0$. This property defines a well-known function space in modern analysis, the so-called *Besov space* $B_p^{s,\infty}$ (e.g., see ref. 4). Furthermore, we shall show that the formula (3) holds when the fractal dimension $D(h)$ is taken to be a certain "box-counting dimension" $D_B(h)$ (assumed to exist) and z_p is defined in terms of the maximal Besov index of order p, s_p, as $z_p = s_p \cdot p$. Although some special examples are known,[5] the previous condition is the only general criterion we know for validity of the "multifractal formalism." The result has great significance for attempts at physical theories of the exponents z_p based upon use of Eq. (3), such as that of She and Leveque,[6] since it is then important to know for what dimension spectrum $D(h)$—if any—that formula holds. Recently, counterexamples have been constructed by Jaffard[7] showing that the formula cannot be generally valid with either the Hausdorff dimension $D_H(h)$ or the usual fractal (box-counting) dimension $D_F(h)$. Our results do not directly address the question of validity of the "multifractal model" of turbulence, but they do show that this issue is substantially equivalent to the question whether turbulent solutions of incompressible Euler equations possess Besov space regularity. This latter issue shall be discussed further in the conclusion section of this paper.

For now, it is helpful to indicate a few consequences of the Parisi–Frisch formula for z_p which shall be established as theorems below. First, since $D(h) \geqslant 0$ for all h, we see that

$$\begin{aligned} \frac{z_p - d}{p} &= \inf_{h \in [h_{\min}, h_{\max}]} \left[h - \frac{D(h)}{p} \right] \\ &\leqslant h_{\min} \end{aligned} \tag{6}$$

Therefore, the minimum Hölder exponent $h_{\min}$ must be no smaller than the largest of $(z_p - d)/p$ for all p. Second, since $z_p < ph + (d - D(h))$ for all h, it follows therefore that

$$D(h) \leqslant ph + (d - z_p) \tag{7}$$

so that there are bounds for the fractal dimensions of "singularity sets" in terms of the exponents z_p for each p. We should mention that bounds of this sort were recently established for Hausdorff dimensions $D_H(h)$ by Jaffard[8] and our analysis was primarily motivated by his work. It is worth remarking that, in fact,

$$D(h) = \inf_p \, [ph + (d - z_p)] \tag{8}$$

under the assumption that $D(h)$ is convex [and otherwise the right-hand side gives the convex hull, conv(D)].

Let us outline now the contents of this paper. In the following Section 2 we shall review some of the fundamentals of the theory of Besov spaces that we shall use in our proofs. Particularly important for us are characterizations of those spaces by orthonormal wavelet bases. In this section we shall also discuss the problem of defining local Hölder classes of negative index (which has been considered as well in the recent work of Jaffard.[7]) With the definition we adopt, all of the results of the paper carry over to the situation with "negative Hölder singularities" with no change in the proofs. Finally, we establish in this section the relation between the "multiscaling exponent" and the maximal Besov index of order p. Our discussion in this section already establishes the weak-sense "multifractality" of Besov space distributions. However, the main new results are established in Sections 3 and 4. In particular, we shall extend the result of Jaffard by proving that his upper bound holds also for $s < d/p$, allowing for "negative Hölder exponents." In addition, we show that the maximal Besov index is given by the Legendre transform of the codimension for a certain "box-counting dimension," whenever the latter exists. The concluding Section 4 contains our discussions on consequences for turbulence. We shall consider there briefly the issue of Besov space regularity for turbulent solutions of Euler equations. Also, we discuss some distinctions between geometric and random multifractal functions and the issue whether "multiscaling" indeed holds for turbulent velocity fields.

2. BESOV SPACES AND MULTISCALING FUNCTIONS

2.1. Reprise of Basic Definitions and Properties

The Besov spaces $B_p^{s,q}$ for $s \in \mathbf{R}, 0 < p, q \leqslant +\infty$ are now ordinarily defined by a "scale decomposition" in terms of a smooth partition of unity in the Fourier representation.[4] For simplicity we shall discuss in this section mostly the case of distributions on Euclidean space $\mathbf{R}^d$, but the theory extends to distributions on the torus $\mathbf{T}^d$.[9] We likewise restrict ourselves to scalar functions, because the whole theory carries over to vector-valued functions with only minor modifications. By a smooth decomposition of scale we mean a set of functions $\{\phi_N : N \geqslant 1\}$, with each ϕ_N in the Schwartz space $\mathscr{S}(\mathbf{R})$ such that

$$\operatorname{supp} \phi_N \subseteq \{k : |k| \in [2^{N-1}, 2^{N+1}]\} \tag{9}$$

for $N > 1$,

$$\operatorname{supp} \phi_1 \subseteq \{k : |k| \in [0, 2]\} \tag{10}$$

and

$$\sum_{N \geqslant 1} \phi_N = 1 \tag{11}$$

For any Schwartz class distribution $f \in \mathscr{S}'(\mathbf{R}^d)$, its Nth-scale component f_N may be defined as

$$f_N(\mathbf{r}) = \frac{1}{(2\pi)^d} \langle \hat{f}(\mathbf{k}), \phi_N(|\mathbf{k}|)\, e^{i\mathbf{k}\cdot\mathbf{r}} \rangle \tag{12}$$

in terms of the Fourier transform distribution $\hat{f}(\mathbf{k})$. Here, $\langle f, \phi \rangle$ denotes the usual canonical evaluation of a distribution f on an element ϕ of the Schwartz class of test functions. Observe that f_N is, for each N, an entire function of exponential type. The class of distributions $B_p^{s,q}$ is defined as that subspace of $\mathscr{S}'(\mathbf{R}^d)$ such that $(2^{sN} f_N) \in l^q(L^p(\mathbf{R}^d))$, i.e., such that

$$\|f\|_{B_p^{s,q}} = \|f_1\|_{L^p} + \left[\sum_{N>1} (2^{sN} \|f_N\|_{L^p})^q \right]^{1/q} < \infty \tag{13}$$

Here $\|\cdot\|_{L^p}$ denotes the usual L^p-norm. When $p = \infty$ or $q = \infty$, the norms are interpreted in the conventional sense as supremum norms. The space $B_p^{s,q}$ is a Banach space with the norm $\|\cdot\|_{B_p^{s,q}}$ when $1 \leqslant p,\ q \leqslant \infty$ and otherwise a quasi-Banach space. (Note that $\|\cdot\|_{B_p^{s,q}}$ depends upon the smooth scale decomposition $\{\phi_N\}$, but all such decompositions define equivalent norms or quasinorms.) We shall, in fact, be mostly concerned with the space $B_p^s \equiv B_p^{s,\infty}$. Note the elementary embedding result that

$$B_p^{s,q} \subset B_p^{s,q'} \tag{14}$$

for $0 < q < q' \leqslant \infty$, so that the spaces with $q < \infty$ are all contained in B_p^s. Another easily proved embedding is

$$B_p^{s',q'} \subset B_p^{s,q} \tag{15}$$

for $s' > s$ and any q, q' (see Section 2.3.2 of ref. 4).

From the definition given above, it clearly would be very natural to have characterizations of the spaces $B_p^{s,q}$ in terms of wavelets.[10,11] In fact, a variety of such characterizations based on "atomic decompositions" are known, and we shall not discuss all of these (but see refs. 11 and 12). We only mention the result most useful for us here, which characterizes the distribution spaces $B_p^{s,q}$ by compactly-supported, orthonormal wavelet bases. Suppose that ψ_ν for $\nu = 1, ..., 2^d - 1$ are "mother wavelets" and ϕ an associated "scale function" which generate an orthonormal basis (in $L^2(\mathbf{R}^d)$) as

$$\psi_{N\mathbf{n}\nu}(\mathbf{r}) = 2^{dN/2}\psi_\nu(2^N\mathbf{r} - \mathbf{n}) \tag{16}$$

for $N \geqslant 0$ and $\mathbf{n} \in \mathbf{Z}^d$ and

$$\phi_{0,\mathbf{n}}(\mathbf{r}) = \phi(\mathbf{r} - \mathbf{n}) \tag{17}$$

for $\mathbf{n} \in \mathbf{Z}^d$. It is known that there are examples of such ψ_ν, ϕ which are compactly supported and C^k for any choice of positive integer k.[10] Then, it may be shown that a Schwartz distribution f lies in $B_p^{s,q}$ for any $s < k$ if and only if

$$\left(\frac{1}{2^{dN}} \sum_{\mathbf{n} \in \mathbf{Z}^d} |\langle f, \phi_\mathbf{n}\rangle|^p\right)^{1/p} < \infty \tag{18}$$

and

$$\left[\sum_{N \geqslant 0} \left(\frac{1}{2^{dN}} \sum_{\mathbf{n} \in \mathbf{Z}^d, \nu} |2^{N(d/2+s)}\langle f, \psi_{N\mathbf{n}\nu}\rangle|^p\right)^{q/p}\right]^{1/q} < \infty \tag{19}$$

These conditions are modified in the obvious way for $p, q = \infty$. See ref. 11 (also ref. 12 for similar results). The orthonormality is not crucial to our arguments in the following section, but the compact support property is essential.

For $p \geqslant 1$ and $s > 0$ the Besov space $B_p^{s,q}$ has an intuitive significance which is exposed by a theorem on equivalent quasinorms. Indeed, let $\Delta_\mathbf{h}$ be defined as the difference operator on $\mathscr{S}'(\mathbf{R}^d)$ given by $\Delta_\mathbf{h} = S_\mathbf{h} - 1$, where $S_\mathbf{h}$ represents space translation. Then, for any integer $k > s$,

$$\|f\|'_{B_p^{s,q}} = \|f\|_{L^p} + \left(\int_{|\mathbf{h}| \leqslant 1} \frac{d^d\mathbf{h}}{|\mathbf{h}|^{d+sq}} \|\Delta_\mathbf{h}^k f\|_{L^p}^q\right)^{1/q} \tag{20}$$

is an equivalent quasinorm with $\|f\|_{B_p^{s,q}}$ (or norm, for $q \geqslant 1$). See Section 2.5.12 of ref. 4. Therefore, $B_p^{s,q}$ is the subspace of L^p which is Hölder continuous of index s in the space L^p-mean sense in the scale L^q-sense.

In particular, B^s_p for $s>0$, $p\geqslant 1$ is the subspace of L^p which is Hölder continuous in the space L^p-mean sense and B^s_∞ for $s>0$ is the usual Hölder–Lipschitz space C^s (the Zygmund class for integer s).

2.2. Local Hölder Classes of Negative Index

Because the identity $C^s=B^s_\infty$ holds for $s>0$ while the Besov spaces are defined for all real s, it is natural to define Hölder classes of negative index to preserve this identity. Because of the cited theorems on wavelet characterization of the Besov spaces it follows that $f\in\mathscr{S}'(\mathbf{R}^d)$ is in C^s, for any real s, if and only if for some $C>0$

$$|\langle f, \phi_{0,\mathbf{n}}\rangle|\leqslant C \tag{21}$$

for all $\mathbf{n}\in\mathbf{Z}^d$, and

$$|\langle f, \psi_{N\mathbf{n}\nu}\rangle|\leqslant C\cdot 2^{-N(d/2+s)} \tag{22}$$

for all $N\geqslant 1$, $\mathbf{n}\in\mathbf{Z}^d$, $\nu=1,\ldots,2^d-1$. This result extends well-known characterizations of the class C^s for $s>0$. It is easy to check that, when $s>-d$, a function of the type

$$f(\mathbf{r})=|\mathbf{r}|^s\,\phi(\mathbf{r}) \tag{23}$$

with ϕ a C^∞ function of compact support, $=1$ in a neighborhood of the origin, satisfies the previous criterion and therefore belongs to C^s with our definition. Therefore, this definition of C^s captures some of our intuitive concept of a function with a "negative Hölder singularity."

However, one would also like to have a definition of a local Hölder class $C^s(\mathbf{r})$ of negative index. For $s>0$, $C^s(\mathbf{r})$ consists of those functions $f\in L^\infty$ such that

$$\sup_{|\mathbf{h}|\leqslant 1}\left(\frac{|f(\mathbf{r}+\mathbf{h})-f(\mathbf{r})|}{|\mathbf{h}|^s}\right)<\infty \tag{24}$$

Parisi and Frisch[1] proposed to define $C^s(\mathbf{r})$ for $s<0$ by the same condition but without the subtraction of $f(\mathbf{r})$ (which now $=\infty$). However, this definition is not natural in the context of our definition of C^s for $s<0$, since it implies that f must be finite in the closed ball around $\mathbf{r}$ excluding its center. In fact, there are elements of B^s_∞ for $s<0$ which are infinite on a countable dense set of points. A simple example was given by Jaffard,[8] which is defined in terms of the function f in Eq. (23) and an enumeration $(\mathbf{q}_k : k\geqslant 1)$ of the rational points of $\mathbf{R}^d$, as

$$g=\sum_{k\geqslant 1}\frac{1}{k^2}f(\cdot-\mathbf{q}_k) \tag{25}$$

This series obviously converges in the norm $\|\cdot\|_{B^s_\infty}$, but the element $g \in B^s_\infty$ so defined is infinite at every rational point.

Instead, we propose a definition motivated by the Theorem 9.2.1 in ref. 10 (which simplified an earlier result of Jaffard.[13]) It was established there that if $f \in C^s(\mathbf{r})$ for $s > 0$, then for a compactly-supported, orthonormal wavelet basis C^k of order $k > s$,

$$\sup_{\{\mathbf{n}, \nu : \mathrm{supp}\, \psi_{N\mathbf{n}\nu} \ni \mathbf{r}\}} |\langle f, \psi_{N\mathbf{n}\nu} \rangle| \leqslant C \cdot 2^{-N(d/2+s)} \tag{26}$$

for all $N \geqslant 1$. Conversely, if Eq. (26) holds and $f \in C^\varepsilon$ for some $\varepsilon > 0$, then

$$|f(\mathbf{r}+\mathbf{h}) - f(\mathbf{r})| \leqslant C\, |\mathbf{h}|^s \log\left(\frac{2}{|\mathbf{h}|}\right) \tag{27}$$

for all $\mathbf{h} \in \mathbf{R}^d$. This is not an exact equivalence, and it is known that the condition $f \in C^\varepsilon$ and the logarithmic correction are necessary. However, we propose to make the following definition.

Definition 1. For $s \in \mathbf{R}$, $f \in C^s[\mathbf{r}] \Leftrightarrow$ for some $C > 0$,

$$\sup_{\{\mathbf{n}, \nu : \mathrm{supp}\, \psi_{N\mathbf{n}\nu} \ni \mathbf{r}\}} |\langle f, \psi_{N\mathbf{n}\nu} \rangle| \leqslant C \cdot 2^{-N(d/2+s)} \qquad \text{for all} \quad N \geqslant 1$$

Our attitude here is somewhat pragmatic. It would be better to make a definition which coincides with the usual one when $s > 0$. However, for $s > 0$ this definition is "nearly" equivalent to the usual one. Furthermore, with this definition $f \in C^s$ if and only if $f \in C^s[\mathbf{r}]$ for all $\mathbf{r} \in \mathbf{R}^d$ with a uniform constant C, for all $s \in \mathbf{R}$. In particular, the example g defined in Eq. (25) is in $C^s[\mathbf{r}]$ (for every $\mathbf{r} \in \mathbf{R}^d$). Let us observe that for our proof in the following section it is enough to have only the forward implication that $f \in C^s[\mathbf{r}]$ implies

$$\sup_{\{\mathbf{n}, \nu : \mathrm{supp}\, \psi_{N\mathbf{n}\nu} \ni \mathbf{r}\}} |\langle f, \psi_{N\mathbf{n}\nu} \rangle| \leqslant C \cdot 2^{-N(d/2+s)} \qquad \text{for all} \quad N \geqslant 1$$

Finally, as remarked earlier, the issue of defining negative Hölder classes has also been discussed at some length in a very recent work of Jaffard.[7]

2.3. Multiscaling Functions and Maximal Besov Classes

Formalizing our earlier considerations, we say that a function $f \in L^p$ is "scaling of order p" if

$$\lim_{|\mathbf{h}| \to 0} \frac{\log \|\Delta_{\mathbf{h}} f\|_p^p}{\log |\mathbf{h}|} = z \tag{28}$$

for some $z>0$, and is "multiscaling" if $f\in L^\infty$ and if Eq. (28) holds for some $z_p>0$ for every $p\geqslant 1$. It is known that the class of such functions is nonempty.[5] Let us now define "maximal Besov classes" $\bar{B}_p^{s,q}$ as

$$\bar{B}_p^{s,q}\equiv\bigcap_{s'<s}B_p^{s',q}\Big\backslash\bigcup_{s'>s}B_p^{s',q} \tag{29}$$

It is straightforward to check that a function scaling of order p with scaling exponent z lies in $\bar{B}_p^s$ for $s=z/p$. In fact, for every $\varepsilon>0$,

$$\frac{\log\|\Delta_{\mathbf{h}}f\|_p^p}{\log|\mathbf{h}|}>(s-\varepsilon)p \tag{30}$$

for sufficiently small $|\mathbf{h}|$, which implies $f\in B_p^{s-\varepsilon}$. On the other hand, if it were true that $f\in B_p^{s+\varepsilon}$ for $\varepsilon>0$, then for small enough $|\mathbf{h}|$,

$$\frac{\log\|\Delta_{\mathbf{h}}f\|_p^p}{\log|\mathbf{h}|}>(s+\varepsilon)p+\frac{p\cdot\log C}{\log|\mathbf{h}|} \tag{31}$$

which would contradict Eq. (28).

Some interesting conclusions can be drawn from this observation by recalling a fundamental embedding result for Besov spaces, namely, $B_p^{s,q}\subset B_{p'}^{s',q}$ (continuous embedding) if $p'>p$ and $s-d/p=s'-d/p'$. See Section 2.7.1 of ref. 4. Obviously $s'<s$. In particular, $B_p^s\subset C^{s-d/p}$, for all $s\in\mathbf{R}$ with our extended definition of the Hölder classes, and elements of B_p^s are Hölder continuous in the usual sense if $s>d/p$. Immediately, a scaling function of order p with exponent z has generalized Hölder exponent at least $(z-d)/p$, and is Hölder continuous in the classical sense if $z>d$. Furthermore, a multiscaling function has generalized Hölder exponent no less than $h_p\equiv(z_p-d)/p$ for any $p\geqslant 1$. This result is exactly what was concluded in Eq. (5) of the introduction by means of the Parisi–Frisch expression for z_p. Let us note, furthermore, that h_p must be a nondecreasing function of p, since the embedding theorem implies for $p'>p$ that $f\in\bar{B}_p^{s_p}\subset B_{p'}^{s'-\varepsilon}$ for all $\varepsilon>0$ and for s' such that $s'-d/p'=s_p-d/p=h_p$. However, it then follows that $s_{p'}\geqslant s'$ and $h_{p'}=s_{p'}-d/p'\geqslant h_p$. Therefore, the limit

$$h_{\min}=\lim_{p\to+\infty}h_p \tag{32}$$

exists (possibly $=+\infty$) and gives the minimum Hölder singularity of f. This result may also be stated as

$$z_p\sim h_{\min}\cdot p+o(p) \tag{33}$$

for $p \to +\infty$, a well-known result of the multifractal formalism, which is here established independently. This fact has some theoretical importance, since it shows that the K41 prediction $z_p = p/3$ can only be true if $h_{\min} = 1/3$, which seems dynamically unlikely. Furthermore, the embedding theorems cited show that any element of B_p^s for any real s and $p > 0$ is "multifractal" in the weak sense, since $B_p^s \subset C^{s-d/p}$, with our generalized definition, and every point $\mathbf{r} \in \mathbf{R}^d$ belongs to $C^h[\mathbf{r}]$ for some $h \geqslant s - d/p$. We may define $S(h)$ to be the set of points where the maximal Hölder exponent is $<h$, and this set may always be assigned Hausdorff, (upper and lower) fractal dimensions, etc. Therefore, a part of the "multifractal" description follows immediately from Besov regularity.

3. THE MULTIFRACTAL FORMALISM

3.1. Upper Bound on the Hausdorff Dimension of $S(h)$

Although we have already seen that a form of the "multifractal hypothesis" is correct under just the assumption of Besov regularity, we have still to derive— and state the conditions of applicability of—the formulas for scaling exponents, fractal dimensions, etc., proposed by Parisi and Frisch. As a first step we shall in this section prove an upper bound on the Hausdorff dimension of the set $S(h)$ when $f \in B_p^s$. To avoid technical complications, we shall give the proofs for distributions on the torus $\mathbf{T}^d$, but it is easy to carry out the extension to $\mathbf{R}^d$ with a little work. Note that the compactly-supported wavelets are easily "periodized" to give an orthonormal basis of $L^2(\mathbf{T}^d)$ and our considerations of the previous section all apply.[10] In this case there are only $2^{dN}(2^d - 1)$ wavelets for each $N \geqslant 1$ (which is the main simplification of working on $\mathbf{T}^d$).

Let us make precise here our definition of the "singularity sets" $S(h)$. For every point $\mathbf{r} \in \mathbf{T}^d$, we may define maximal Hölder classes at that point as

$$\bar{C}^h[\mathbf{r}] = \bigcap_{h' < h} C^{h'}[\mathbf{r}] \Big\backslash \bigcup_{h' > h} C^{h'}[\mathbf{r}] \tag{34}$$

and, for every $f \in B_p^s$, we may then set $\underline{h}[\mathbf{r}] = h$ if and only if $f \in \bar{C}^h[\mathbf{r}]$. The definition of "singularity set" that we use in our argument is then

$$S(h) = \{\mathbf{r} : \underline{h}[\mathbf{r}] < h\} \tag{35}$$

We denote as $D_{\mathrm{H}}(h) = \dim_{\mathrm{H}} S(h)$ the Hausdorff dimension of $S(h)$. For a readable discussion of the definitions and relations of the various dimensions we consider, see ref. 14.

We now prove the following:

Theorem 1. If $f \in B^s_p$ for $s \in \mathbf{R}$ and $p > 0$, then for $z = s \cdot p$,

$$D_{\mathrm{H}}(h) \leqslant ph + (d - z) \tag{36}$$

for all $h > s - (d/p)$.

Proof. For each $N \geqslant 1$, set

$$S_N(h) \equiv \bigcup_{\mathbf{n}\nu} \{\mathrm{supp}\, \psi_{N\mathbf{n}\nu} : |\langle \psi_{N\mathbf{n}\nu}, f \rangle| > C \cdot 2^{-(d/2+h)N}\} \tag{37}$$

for some $C > 0$, and

$$S^*(h) \equiv \limsup_N S_N(h) \tag{38}$$

Therefore, for every $\mathbf{r} \in (\mathbf{T}^d \backslash S^*(h))$, it follows that for all N sufficiently large and for all $\mathbf{n}$, ν, that $\mathbf{r} \in \mathrm{supp}\, \psi_{N\mathbf{n}\nu}$ implies $|\langle \psi_{N\mathbf{n}\nu}, f \rangle| \leqslant C \cdot 2^{-(d/2+h)N}$. In other words, $\underline{h}[\mathbf{r}] \geqslant h$, so that

$$S(h) \subseteq S^*(h) \tag{39}$$

It suffices to bound the Hausdorff dimension of $S^*(h)$.

Recall that $f \in B^s_p$ is equivalent to

$$\sup_{N \geqslant 1} \left(\frac{1}{2^{dN}} \sum_{\mathbf{n}\nu} |2^{(d/2+s)N} \langle f, \psi_{N\mathbf{n}\nu} \rangle|^p \right) < +\infty \tag{40}$$

Clearly, this requires that

$$\#\{(\mathbf{n}, \nu) : |\langle \psi_{N\mathbf{n}\nu}, f \rangle| > C \cdot 2^{-(d/2+h)N}\} \leqslant (\mathrm{const})\, 2^{(hp+(d-z))N} \tag{41}$$

Since $\mathrm{diam}(\mathrm{supp}\, \psi_{N\mathbf{n}\nu}) = 2^{-N}R$, for some $R > 0$, it follows that $S_N(h)$ can be covered by just $O(2^{(hp+(d-z))N})$ closed balls of diameter $\delta_N \equiv 2^{-N}R$. Thus, for $D > hp + (d - z)$,

$$\begin{aligned} \mathscr{H}^D_{\delta_N}(\limsup_{N'} S_{N'}) &\leqslant \mathscr{H}^D_{\delta_N}\left(\bigcup_{N' \geqslant N} S_{N'} \right) \\ &\leqslant (\mathrm{const}) \sum_{N' \geqslant N} 2^{[hp+(d-z)]N'} (2^{-N'}R)^D \\ &\to 0 \end{aligned} \tag{42}$$

as $N \to +\infty$. Therefore,

$$\dim_H (S^*(h)) \leqslant hp + (d-z) \quad \blacksquare \tag{43}$$

The proof of this theorem is very similar to the argument made by Jaffard,[8] but somewhat simpler. However, it extends his result to the case $s < d/p$, allowing for "negative exponents." Note that if $s > d/p$ we recover essentially Jaffard's original result. The reason is that, for that case, $f \in C^\varepsilon$, with $\varepsilon = s - (d/p) > 0$, so that $f \in C^h[\mathbf{r}]$ implies that for all $\mathbf{r}'$, $|\mathbf{r} - \mathbf{r}'| \leqslant 1$,

$$\begin{aligned} |f\mathbf{r}) - f(\mathbf{r}')| &\leqslant C \cdot |\mathbf{r} - \mathbf{r}'|^h \log\left(\frac{2}{|\mathbf{r} - \mathbf{r}'|}\right) \\ &\leqslant C\, |\mathbf{r} - \mathbf{r}'|^{h-\varepsilon} \end{aligned} \tag{44}$$

for all $\varepsilon > 0$. Therefore, $\underline{h}(\mathbf{r}) \geqslant h$, where $\underline{h}$ is defined as above but using the conventional definition of local Hölder classes. In other words, under the additional assumption that $f \in C^\varepsilon$, $\underline{h}[\mathbf{r}] \geqslant h$ implies that $\underline{h}(\mathbf{r}) \geqslant h$, and

$$\{\mathbf{r} : \underline{h}(\mathbf{r}) < h\} \subseteq S(h) \tag{45}$$

Jaffard bounded the Hausdorff dimension of a slightly larger set $\{\mathbf{r} : f \text{ not in } C^h(\mathbf{r})\}$, for the case $s > (d/p)$.

It is a consequence of Theorem 1 that, in fact, whenever $f \in B^s_p$,

$$z = \inf_{h > s - (d/p)} [hp + (d - D_{\mathrm{H}}(h))] \tag{46}$$

since the infimum is always achieved at the lower bound $h_p = s - (d/p)$, because $D_{\mathrm{H}}(h_p) = 0$. This result already appears close to the Parisi–Frisch formula (3). However, notice that, for a given p, this formula holds for *every* s such that $f \in B^s_p$. In particular, if it holds for one s, it also holds for any $s' < s$. Furthermore, in the Parisi–Frisch formula for the scaling exponent z_p the infimum should be taken only over $h \geqslant h_{\min}$. To prove this stronger result, we need to use the further information that $s_p = z_p/p$ is the *maximal* Besov index of order p.

3.2. Parisi–Frisch Formula for the Maximal Besov Index

We have already cited the work of Daubechies and Lagarias[5] which establishes the validity of the Parisi–Frisch formula for the maximal Besov index of the solutions of some functional "refinement equations." However, in a more recent work[7] Jaffard has constructed a simple counterexample which shows that the Parisi–Frisch formula cannot be generally valid (in particular, as a lower bound) if the "dimension spectrum" $D(h)$ appearing there is interpreted as either the Hausdorff dimension or box-counting dimension of the singularity sets $S(h)$. Nevertheless, we shall show here that there is an interpretation of the dimension spectrum for which the formula has a general validity in the Besov space context. It may be noted that our approach is very similar to the simple multifractal formalism for measures which was proposed by Falconer in Chapter 17 of ref. 14.

To set up the formalism, we require some definitions. First, we introduce, for each $f \in \mathcal{S}'(\mathbf{T}^d)$, "normalized" wavelet coefficients

$$A_{N\mathbf{n}\nu} \equiv 2^{dN/2} \langle f, \psi_{N\mathbf{n}\nu} \rangle \tag{47}$$

where $\{\psi_{N\mathbf{n}\nu} : N\mathbf{n}\nu\}$ is an orthonormal wavelet basis. In fact, neither compact support nor orthonormality will be required here, and it is possible to take $\{\psi_{N\mathbf{n}\nu}\}$ to be any Schwartz class "atomic decomposition" for which criteria like Eqs. (18) and (19) hold. Then, for each $h \in \mathbf{R}$, $\eta > 0$, and $N \geqslant 1$, let

$$\mathcal{N}_{N,\eta}(h) \equiv \sharp\{\mathbf{n}\nu : 2^{-(h-\eta)N} \geqslant |A_{N\mathbf{n}\nu}| > 2^{-(h+\eta)N}\} \tag{48}$$

We can then define upper and lower "box-counting dimensions" as

$$\bar{D}_{\mathrm{B}}(h) \equiv \lim_{\eta \downarrow 0} \limsup_{N \to +\infty} \frac{\log_2 \mathcal{N}_{N,\eta}(h)}{N} \tag{49}$$

and

$$\underline{D}_{B}(h) \equiv \lim_{\eta \downarrow 0} \liminf_{N \to +\infty} \frac{\log_2 \mathcal{N}_{N,\eta}(h)}{N} \tag{50}$$

It is easy to see that these quantities are well-defined (possibly $= -\infty$) since $\mathcal{N}_{N,\eta}(h)$ is nondecreasing in η for each fixed N, h. An important cautionary remark is that these "box-counting dimensions" will not generally be the dimensions—box-counting or otherwise—of any particular space sets such as $S(h)$. Instead, they give information simply about the number of wavelet coefficients of a certain magnitude but without providing any information on the location of the corresponding wavelets. Therefore, "large" coefficients at one scale of resolution may be located in entirely different regions of space than those at another scale.

Although these quantities are not the dimensions of any subsets of space, nevertheless they have a number of the properties one would reasonably expect. We collect these into the following:

Proposition 1. For all $h \in \mathbf{R}$: (i) $\underline{D}_{\mathrm{B}}(h) \leqslant \bar{D}_{\mathrm{B}}(h) \leqslant d$; (ii) $\underline{D}_{\mathrm{B}}(h) > -\infty$ implies $\underline{D}_{\mathrm{B}}(h) \geqslant 0$ (and the same property for $\bar{D}_{\mathrm{B}}$); (iii) $\underline{D}_{\mathrm{B}}(h) = \bar{D}_{\mathrm{B}}(h) = -\infty$ for $h < h_{\min}$.

Proof (i) The first inequality $\underline{D}_{\mathrm{B}}(h) \leqslant \bar{D}_{\mathrm{B}}(h)$ is immediate from the definitions. Also, since $\mathcal{N}_{N,\eta}(h) \leqslant (\text{const})\, 2^{dN}$ (the total number of wavelets at scale N), the last part $\bar{D}_{\mathrm{B}}(h) \leqslant d$ follows easily. (ii) These results are direct consequences of the definitions and the fact that $\mathcal{N}_{N,\eta}(h) > 0$ only if $\mathcal{N}_{N,\eta}(h) \geqslant 1$. (iii) If $h < h_{\min}$, then $h + 2\eta_0 < h_{\min}$ for some $\eta_0 > 0$. It then follows from the fact that $|A_{N\mathbf{n}\nu}| \leqslant (\text{const})\, 2^{-(h_{\min} - \eta_0)N}$ for all N, $\mathbf{n}$, ν that $\mathcal{N}_{N,\eta}(h) = 0$ for all $\eta < \eta_0$ and all N sufficiently large. This gives the result. ∎

The second part of this proposition states that the "box-counting dimensions" can only be negative in the rather trivial case where they are equal to minus infinity. The third part expresses the lack of any singularities worse than $h_{\min}$.

Making use of these properties, we now prove the following:

Theorem 2. Let $f \in \bar{B}^s_p$ for $p > 0$. If $z_p = s \cdot p$, then

$$z_p \geqslant \inf_{h \geqslant h_{\min}} [ph + (d - \bar{D}_{\mathrm{B}}(h))] \tag{51}$$

Proof. Let us *define*

$$\begin{aligned} z_p^* &\equiv \inf_{h \geqslant h_{\min}} [ph + (d - \bar{D}_{\mathrm{B}}(h))] \\ &\leqslant ph_{\min} + d < +\infty \end{aligned} \tag{52}$$

and $s^* = z_p^*/p$. We must prove that $z_p \geqslant z_p^*$. In fact, we show, for any $\varepsilon > 0$, that $f \in B_p^{s^* - \varepsilon}$ which implies $z_p^* \leqslant z_p$. Set $s' \equiv s^* - \varepsilon$. We show that for N_0 large enough,

$$\sup_{N \geqslant N_0} \left(\frac{1}{2^{dN}} \sum_{\mathbf{n}\nu} |2^{(d/2 + s')N} \langle f, \psi_{N\mathbf{n}\nu} \rangle|^p \right) < +\infty \tag{53}$$

which is equivalent to $f \in B_p^{s^* - \varepsilon}$. It follows from the definition of "box-counting dimensions" that for each $h \in [h_{\min}, s^*]$ and $\varepsilon > 0$, there exists an $\eta_0(h) < \varepsilon/2$ and an integer $N_0(h)$ such that

$$\mathcal{N}_{N,\eta}(h) \leqslant 2^{(\bar{D}_{\mathrm{B}}(h) + \varepsilon p/2)N} \tag{54}$$

when $\eta < \eta_0(h)$ and $N \geqslant N_0(h)$. Because the compact interval $[h_{\min}, s^*]$ is covered by such open intervals $(h - \eta_0(h), h + \eta_0(h))$, it is possible to select a finite subcollection $(h_i - \eta_i, h_i + \eta_i)$, $i = 1, ..., K$, which is still a covering:

$$[h_{\min}, s^*] \subseteq \bigcup_{i=1}^{K} (h_i - \eta_i, h_i + \eta_i) \tag{55}$$

Defining $N_0 = \max_{1 \leqslant i \leqslant K} N_0(h_i)$, it follows also that

$$\mathcal{N}_{N,\eta_i}(h_i) \leqslant 2^{(\bar{D}_B(h_i) + \varepsilon p/2)N} \tag{56}$$

for each $i = 1, \ldots, K$, if $N \geqslant N_0$. Thus,

$$\begin{aligned} \sup_{N \geqslant N_0} \left(\frac{1}{2^{dN}} \sum_{\mathbf{n}\nu} |2^{s'N} A_{N\mathbf{n}\nu}|^p \right) &\leqslant \sup_{N \geqslant N_0} \left[\frac{1}{2^{dN}} \sum_{i=1}^{K} 2^{[s' - (h_i - \eta_i)]pN} 2^{N(\bar{D}_B(h_i) + \varepsilon p/2)} + 2^{(s' - s^*)pN} \right] \\ &\leqslant \sup_{N \geqslant N_0} \left[\sum_{i=1}^{K} 2^{\{z^* - [h_i p + (d - \bar{D}_B(h_i))]\} N} 2^{(\eta_i - \varepsilon/2)pN} + 2^{-\varepsilon \cdot pN} \right] \\ &\leqslant K + 1, \end{aligned} \tag{57}$$

since the choice was made for each i that $\eta_i < \varepsilon/2$. ∎

If one defines a "wavelet structure function" $S_p^{(N)}$ as

$$S_p^{(N)} \equiv \frac{1}{2^{dN}} \sum_{\mathbf{n}\nu} |A_{N\mathbf{n}\nu}|^p \tag{58}$$

then it is a straightforward consequence of the definition of limit-infimum and the criterion (19) that the maximal Besov index of order p, z_p, may be expressed as

$$z_p = \liminf_{N \to +\infty} \frac{-\log_2 S_p^{(N)}}{N} \tag{59}$$

Immediately, there is an upper bound

$$z_p \leqslant \limsup_{N \to +\infty} \frac{-\log_2 S_p^{(N)}}{N} \tag{60}$$

This formula yields for the maximal Besov index the following estimate from above:

Theorem 3. Let $f \in \bar{B}_p^s$ for $p > 0$. If $z_p = s \cdot p$, then

$$z_p \leqslant \inf_{h \geqslant h_{\min}} [ph + (d - \underline{D}_B(h))] \tag{61}$$

Proof. Consider any $h \geqslant h_{\min}$. For each $\varepsilon > 0$, it follows from the definition of the limit-infimum that one can find an $\eta_0 < \varepsilon/p$ and an N_0 such that

$$\mathcal{N}_{N,\eta}(h) \geqslant 2^{N(\underline{D}_B(h) - \varepsilon)} \tag{62}$$

when $\eta < \eta_0$ and $N \geqslant N_0$. Therefore, for such $N \geqslant N_0$,

$$\begin{aligned} S_p^{(N)} &\geqslant \frac{1}{2^{dN}} \cdot 2^{-(h+\eta_0)pN} \cdot 2^{N(\underline{D}_{\mathrm{B}}(h)-\varepsilon)} \\ &= 2^{-[hp+(d-\underline{D}_{\mathrm{B}}(h))]N} \cdot 2^{-2\varepsilon N} \end{aligned} \tag{63}$$

It follows that

$$\limsup_{N \to +\infty} \frac{-\log_2 S_p^{(N)}}{N} \leqslant [hp + (d - \underline{D}_{\mathrm{B}}(h))] + 2\varepsilon \tag{64}$$

and, since $h \geqslant h_{\min}$ and $\varepsilon > 0$ were arbitrary,

$$\limsup_{N \to +\infty} \frac{-\log_2 S_p^{(N)}}{N} \leqslant \inf_{h \geqslant h_{\min}} [hp + (d - \underline{D}_{\mathrm{B}}(h))] \quad \blacksquare \tag{65}$$

Combining the statements of Theorems 2 and 3, we see, under the same conditions, that

$$\inf_{h \geqslant h_{\min}} [ph + (d - \underline{D}_{\mathrm{B}}(h))] \geqslant z_p \geqslant \inf_{h \geqslant h_{\min}} [ph + (d - \underline{D}_{\mathrm{B}}(h))] \tag{66}$$

In particular, we have the following result.

Corollary 1. If $\underline{D}_{\mathrm{B}}(h) = \bar{D}_{\mathrm{B}}(h)$ for all $h \geqslant h_{\min}$, then for every $p > 0$,

$$\begin{aligned} z_p &= \inf_{h \geqslant h_{\min}} [ph + (d - D_{\mathrm{B}}(h))] \\ &= \lim_{N \to +\infty} \frac{-\log_2 S_p^{(N)}}{N} \end{aligned} \tag{67}$$

In particular, the latter limit exists.

We therefore see that, under certain conditions, the Parisi–Frisch formula gives the maximal Besov index of a distribution f with pth-order Besov regularity. These results automatically include the cases where f is a scaling function of order p with exponent z, and multiscaling of general order. In fact, we see from the last result that the equality of "box-counting dimensions," $\underline{D}_{\mathrm{B}}(h) = \bar{D}_{\mathrm{B}}(h)$ for all h, is a sufficient "multifractal" condition to imply "multiscaling" of the wavelet structure functions of positive order. It is unknown to us whether the condition that

$$\lim_{\eta \downarrow 0} \lim_{N \to +\infty} \frac{\log_2 \mathcal{N}_{N,\eta}(h)}{N} = D_{\mathrm{B}}(h) \tag{68}$$

for all $h \geqslant h_{\min}$ implies that f is "multiscaling" in the sense of differences (28). At least it will then be true that

$$\liminf_{|\mathbf{h}| \to 0} \frac{\log \|\Delta_{\mathbf{h}} f\|_p^p}{\log |\mathbf{h}|} = z_p, \tag{69}$$

with pth-order exponents given by the Parisi–Frisch formula, $z_p = \inf_h [hp + (d - D_{\mathrm{B}}(h))]$, as the consequence of equivalence of quasinorms. One problem which we have not addressed, which should be considered, is the possible "basis dependence" of our definition of $D_{\mathrm{B}}(h)$. It has not been shown that those quantities for a particular Besov distribution f are independent of the atomic decomposition selected in Eq. (48) and therefore it is not clear that they have an intrinsic significance (except when Eq. (8) is valid).

4. IMPLICATIONS FOR TURBULENCE

4.1. Besov Regularity for Turbulent Solutions of Euler equations?

As stated earlier, our results here have no relation to the question whether the "multiscaling" which has been reported experimentally actually occurs in the inertial range of turbulent fluids. We have given elsewhere theoretical arguments using renormalization group and operator-product expansion to establish "multiscaling,"[15,16] but those arguments require hypotheses which are themselves unsubstantiated. (Furthermore, the "multiscaling" is there established for velocity differences without absolute values and for ensemble averages rather than space averages: see further below.) The relevance of the present work is that it shows that multifractality of the velocity fields, i.e., the "multifractal model," follows mathematically under even much weaker conditions than the "multiscaling" which originally motivated Parisi and Frisch to their conjecture. In particular, L^p-mean Hölder continuity, or Besov regularity, implies multifractality in at least the "weak sense" of local Hölder continuity. We believe that Besov regularity in space of the solutions of the incompressible Navier–Stokes equations is very likely to hold in the limit of zero viscosity. Of course, at present, zero-viscosity limiting solutions are shown to exist only in a much weaker sense of "measure-valued solutions,"[17] but such results are often the first step in proving stronger regularity. We hope that some of the methods based on Littlewood–Paley-type decompositions, used recently to prove Besov regularity for transport equations,[18] may apply. It does seem that the assumption of Besov regularity is compatible with the usual phenomenology of high-Reynolds-number turbulence. We believe that the locality of energy transfer in scale, which we have recently proved under a stronger assumption of Hölder type,[19] actually holds just using some space-mean Hölder property. If that is true, then it is possible to show that finite energy dissipation in the zero-viscosity limit requires maximal Besov indices $s_p \leqslant 1/3$ for $p \geqslant 3$, generalizing a result claimed by Onsager[20] (which corresponds to the case $p = +\infty$.) In other words, energy balance alone should require "multiscaling exponents" of order $p \geqslant 3$ to take values $z_p \leqslant p/3$, the Kolmogorov values.[2] The key physical question is the locality of energy transfer in scale under the Besov regularity condition.

4.2. "Geometric" vs. "Random" Multifractality

A remaining issue to examine is our interpretation of expectations as space averages. Under some assumption of space ergodicity, the average in a large enough volume should be close to the ensemble average, but there are delicate questions of exchange of limit operations involved. In fact, there are some reasons to doubt that space averages (in any finite volume) and ensemble averages will coincide for multiscaling behavior, because there are essential differences between "geometric" multifractal functions associated with space averaging and "random" multifractal functions associated with ensemble averaging. These were first pointed out by Mandelbrot.[22] The concept of a random multifractal function may be formulated within the probabilistic framework of large-deviations theory, as follows: If $\mathbf{v}$ is a velocity field chosen from a random ensemble, then a "scale-l Hölder exponent" H_l, another random variable, may be defined as

$$H_l(\mathbf{r}) = \frac{\sup_{|\mathbf{h}| \leqslant l} \log |\mathbf{v}(\mathbf{r}+\mathbf{h}) - \mathbf{v}(\mathbf{r})|}{\log l} \tag{70}$$

Assuming space homogeneity of the random velocity field, we find that these variables are equal in distribution for all choices of space point $\mathbf{r}$. As $l \downarrow 0$, the variable $H_l(\mathbf{r})$ will take on the limiting value h if, in a given realization, the velocity field has maximal Hölder exponent h at point $\mathbf{r}$. (To define the variable H_l to allow for "negative Hölder exponents," wavelet coefficients should be used, following the ideas proposed earlier in this paper, but we do not require that generality here.) Therefore, over the random ensemble, various values for the limit will occur. The "multifractal

[2] These conjectures have now been rigorously established by P. Constantin *et al.*[21] (private communication).

model" may be formulated as a conjecture that the random variable H_l has the large-deviations property with some rate function $I(h)$, so that

$$\text{Prob}(\{H_l \approx h\}) \sim l^{I(h)} \tag{71}$$

for $l \downarrow 0$. Such a probabilistic interpretation was already proposed in the original Parisi–Frisch paper,[1] and it directly leads to the Legendre transform expression for the scaling exponents z_p, with $d - D_{\text{B}}(h)$ replaced by $I(h)$. A large-deviations formulation very close to ours above has been given independently by Frisch.[23]

However, while this concept of "random" multifractality is obviously quite close in concept to the "geometric" notion, involving space averages, which we have employed in this paper, there are essential differences. Chiefly, as noted by Mandelbrot, $I(h)$ may be greater than d, the space dimension. This can be interpreted geometrically as a "negative dimension" $D_{\text{B}}(h)$. However, $D_{\text{H}}(h)$ and $D_{\text{B}}(h)$ as we have used them in this work must be nonnegative. Therefore, it may be that typical realizations chosen from the random ensemble of turbulent velocity fields may not be geometrically "multiscaling" even if ensemble averages show "multiscaling," or that the space-average and ensemble-average scaling exponents may be distinct.

It may be worth pointing out, however, a few elementary relations between space-average and ensemble-average scaling properties which can be readily derived. First, we observe that "multiscaling" Eq. (1) for *ensemble-averages* implies Besov regularity in space as an almost sure statement, assuming just space-homogeneity of the ensemble. To make the proofs as simple as possible, we consider the case of random velocity fields on the unit torus $\mathbf{T}^d$. In fact, for our conclusions only the following consequence of "multiscaling" is required:

$$\langle |\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})|^p \rangle \leqslant (\text{const})\, l^{\zeta_p(1-\varepsilon/2)} \tag{72}$$

for all $l < 1$, with some finite constant for each $\varepsilon > 0$. Let us state the result as a formal theorem.

Theorem 4. Suppose that Eq. (1), or its consequence Eq. (72), holds with $p \geqslant 1$ for average with respect to a homogeneous ensemble. Then for any $\varepsilon > 0$ it follows that a finite constant $C_\varepsilon > 0$ exists a.s. so that

$$\int_{\mathbf{T}^d} d^d\mathbf{r}\, |\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})|^p \leqslant C_\varepsilon \cdot l^{\zeta_p(1-\varepsilon)} \tag{73}$$

for all $l \leqslant 1$. In particular, $\mathbf{v}$ has almost surely pth-order Besov regularity and a maximal Besov index $s_p \geqslant \zeta_p/p$.

Proof. The proof of Eq. (73) is in two steps: (1) First, we establish it for all $\mathbf{l}_{N,i}=2^{-N}\hat{\mathbf{e}}_i$ almost surely, with $\hat{\mathbf{e}}_i$ an orthonormal basis in $\mathbf{R}^d$ and $N\geqslant 0$ an integer. Set $l_N=2^{-N}$. Using space-homogeneity, it follows from Eq. (72) that

$$\langle X_N\rangle \leqslant (\text{const})\cdot l_N^{\zeta_p\varepsilon/2} \tag{74}$$

for all $N\geqslant 0$, where

$$X_N\equiv l_N^{-\zeta_p(1-\varepsilon)}\sum_{i=1}^{d}\int_{\mathbf{T}^d}|\Delta_{\mathbf{l}_{N,i}}\mathbf{v}|^p \tag{75}$$

Applying the Chebyshev inequality then gives for every integer $M>0$ that

$$\text{Prob}\left(X_N>\frac{1}{M}\right)\leqslant(\text{const})\,M\cdot l_N^{\zeta_p\varepsilon/2} \tag{76}$$

and thus

$$\sum_{N=0}^{\infty}\text{Prob}\left(X_N>\frac{1}{M}\right)<+\infty \tag{77}$$

Therefore the Borel–Cantelli lemma implies that $X_N\to 0$ a.s. as $N\to+\infty$, and, in particular, that there exists a finite constant $C_\varepsilon>0$ (which may depend upon the realization) so that $X_N\leqslant C_\varepsilon$. Equivalently, for a set of realizations of probability one,

$$\int_{\mathbf{T}^d}d^d\mathbf{r}\,|\mathbf{v}(\mathbf{r}+\mathbf{l}_{N,i})-\mathbf{v}(\mathbf{r})|^p\leqslant C_\varepsilon\cdot l_N^{\zeta_p(1-\varepsilon)} \tag{78}$$

for all $N\geqslant 0$, $i=1,\dots,d$. (2) We show that Eq. (78) holds in fact for all $\mathbf{l}\in\mathbf{T}^d$ if it holds for the $\mathbf{l}_{N,i}$. Note first, since the shift operator $S_{\mathbf{a}}$ on the torus is an isometry in $L^p(\mathbf{T}^d)$, that

$$\begin{aligned}\|\Delta_{\mathbf{l}_1+\mathbf{l}_2}\mathbf{v}\|_p&=\|S_{\mathbf{l}_2}\Delta_{\mathbf{l}_1}\mathbf{v}+\Delta_{\mathbf{l}_2}\mathbf{v}\|_p\\&\leqslant\|\Delta_{\mathbf{l}_1}\mathbf{v}\|_p+\|\Delta_{\mathbf{l}_2}\mathbf{v}\|_p\end{aligned} \tag{79}$$

Next, for any $\mathbf{l}=(l^{(1)},\dots,l^{(d)})$ choose the largest component $l^{(m)}$ and fix an integer N so that $2^{-(N+1)}\leqslant l^{(m)}<2^{-N}$. Note, in particular, that $l\geqslant 2^{-(N+1)}$. Since each $l^{(i)}$ has a binary expansion $l^{(i)}=\sum_{J\geqslant N+1}\varepsilon_J^{(i)}2^{-J}$, with bits $\varepsilon_J^{(i)}\in\{0,1\}$, it follows that

$$\mathbf{l}=\sum_{J\geqslant N+1,i}\varepsilon_J^{(i)}\mathbf{l}_{J,i} \tag{80}$$

Because we have assumed $\|\Delta_{l_{J,i}}\mathbf{v}\|_p \leqslant C \cdot 2^{\zeta_p(1-\varepsilon)J/p}$, using Eqs. (79), (80), it follows that

$$\|\Delta_l \mathbf{v}\|_p \leqslant \sum_{J \geqslant N+1} \sum_i \varepsilon_J^{(i)} \|\Delta_{l_{J,i}}\mathbf{v}\|_p \leqslant \frac{C \cdot d}{1-2^{-s}} l^s \tag{81}$$

with $s = \zeta_p(1-\varepsilon)/p$. The last statement of the theorem follows by defining Ω_M to be the set of realizations obeying Eq. (73) with $\varepsilon_M = 1/M$. The required set of probability one is $\Omega' = \bigcap_{M \geqslant 1} \Omega_M$. ▮

It is clear that one cannot hope to show more than the inequality $z_p(\omega) \geqslant \zeta_p$, since an individual realization may always have, with some finite probability, greater regularity than the average in a fixed region. On the other hand, a result in the reverse direction is possible if the constants $C_\varepsilon(\omega)$ in the a.s. bounds

$$\int_{\mathbf{T}^d} d^d\mathbf{r}\, |\Delta_l \mathbf{v}(\mathbf{r}, \omega)|^p \leqslant C_\varepsilon(\omega) \cdot l^{z_p(\omega)(1-\varepsilon)} \tag{82}$$

with $s_p(\omega) = z_p(\omega)/p$ the maximal Besov index for sample point ω, can be chosen so that $\langle C_\varepsilon \rangle < +\infty$ for every $\varepsilon > 0$. The optimal choice of $C_\varepsilon(\omega)$ is $\|\mathbf{v}(\cdot, \omega)\|_{B_p^{s_p(1-\varepsilon)}}$, so this condition may be stated as

$$\langle \|\mathbf{v}\|_{B_p^{s_p(1-\varepsilon)}} \rangle < +\infty \tag{83}$$

for every $\varepsilon > 0$. It then follows easily that

$$\langle |\Delta_l \mathbf{v}|^p \rangle \leqslant \langle C_\varepsilon \rangle \cdot l^{z_p(1-\varepsilon)} \tag{84}$$

where $z_p \equiv \text{ess.inf}_\omega\, z_p(\omega)$. As a consequence of the previous Proposition, $z_p \geqslant \zeta_p$. Since the inequality Eq. (84) for every $\varepsilon > 0$ clearly implies also that $\zeta_p \geqslant z_p$, it follows that $\zeta_p = z_p$, or,

$$\zeta_p \equiv \text{ess.inf}_\omega\, z_p(\omega) \tag{85}$$

Of course, ζ_p/p need not be the maximal Besov index of any velocity fields occurring with positive probability and, from the definition of essential infimum, it follows only that ζ_p can be approached arbitrarily closely for a positive probability set.

To make clear our hypotheses in deriving these results, we are assuming that Prob in Eq. (71) is a probability measure obtained by taking the zero-viscosity limit of suitable stationary measures for the Navier–Stokes dynamics with some external driving forces. In other words, we assume an order of limits $t \to +\infty$, then $\nu \to 0$, and lastly $l \to 0$ in Eq. (71). Each of these limiting assumptions requires of course some justification and may prove ultimately to be false. Stationary measures $\text{Prob}_{(\nu)}$ have been constructed for Navier–Stokes equations in a bounded domain, with any positive viscosity $\nu > 0$ and random forces white noise in time, by Vishik and Fursikov in Theorem XI.2.1 of ref. 25 using the Bogolyubov–Krylov averaging method. However, nothing is known about the zero-viscosity limit of these measures and it has been emphasized to us by U. Frisch (private communication) that the limit, even if it exists, may not be a probability measure, as our argument requires. On the other hand, a viscosity-independent moment condition of the form $\sup_{\nu>0} \langle |\delta_l v|^p \rangle_{(\nu)} < \infty$ for any $p > 1$ is sufficient to guarantee tightness of the family of distributions of $\delta_l v$ for each fixed l and guarantees existence of a limit as $\nu \to 0$ (at least along subsequences) which is also a distribution (e.g., see Lemma II.3.1 in ref. 25). Therefore, the limit could only fail to be a probability measure if the moments themselves diverge as $\nu \to 0$. The final large-deviations hypothesis (71) as $l \to 0$ appears to be very far from rigorous proof, and can only be presently justified by more physical arguments based upon "cascade" ideas or renormalization-group pictures.

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REFERENCES

1. U. Frisch and G. Parisi, On the singularity structure of fully-developed turbulence, in *Turbulence and Predictability of Geophysical Flows and Climate Dynamics*, (North-Holland, Amsterdam, 1985), pp. 84–88.
2. F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. A. Antonia, *J. Fluid Mech.* **140**:63 (1984).
3. E. Aurell *et al.*, *J. Fluid Mech.* **238**:467 (1992).
4. H. Treibel, *Theory of Function Spaces* (Birkhaüser, Basel, 1983).
5. I. Daubechies and J. C. Lagarias, On the thermodynamic formalism for multifractal functions, preprint.
6. Z. S. She and E. Leveque, *Phys. Rev. Lett.* **72**:336 (1994).
7. S. Jaffard, Multifractal formalism for functions, Part I: Results valid for all functions, preprint (1994).
8. S. Jaffard, *C.R. Acad. Sci. Paris, Ser. I* **314**:31 (1992).
9. H.-J. Schmeisser and H. Treibel, *Topics in Fourier Analysis and Function Spaces* (Wiley, Chichester, England, 1987).
10. I. Daubechies, *Ten Lectures on Wavelets* (Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1992).
11. Y. Meyer, *Ondelettes et Opérateurs*, I, II, III (Hermann, Paris, 1990).

12. M. Frazier, B. Jawerth, and G. Weiss, *Littlewood–Paley Theory and the Study of Function Spaces* (American Mathematical Society, Providence, Rhode Island, 1991).
13. S. Jaffard, *C.R. Acad. Sci. Paris Ser. I* **308**:79 (1989).
14. K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications* (Wiley, Chichester, England, 1990).
15. G. L. Eyink, *Phys. Lett. A* **172**:355 (1993).
16. G. L. Eyink, *Phys. Rev. E* **48**:1823 (1993).
17. R. J. DiPerna and A. J. Majda, *Commun. Math. Phys.* **108**:667 (1987).
18. R. J. DiPerna, P. L. Lions, and Y. Meyer, *Ann. Inst. Henri Poincaré* **8**:271 (1991).
19. G. L. Eyink, Energy dissipation without viscosity in ideal hydrodynamics, I. Fourier analysis and local energy transfer, *Physica D*, to appear.
20. L. Onsager, *Nuovo Cimento Suppl.* **6**:279 (1949).
21. P. Constantin, Weinan E, and E. S. Titi, Onsager's conjecture on the energy conservation for solutions of Euler's equations, *Commun. Math. Phys.*, to appear.
22. B. B. Mandelbrot, Two meanings of multifractality, and the notion of negative fractal dimension, in *Chaos/Xaoc. Soviet–American Perspectives on Nonlinear Science*, David K. Campbell, ed. (American Institute of Physics, New York, 1990).
23. U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
24. R. Benzi *et al.*, Random process for the construction of multiaffine fields, *Physica D* (1993), in press.
25. M. J. Vishik and A. V. Fursikov, *Mathematical Problems of Statistical Hydrodynamics* (Kluwer, Dordrecht, 1988).